ON CONSTRUCTIONS OF SOME CLASSES OF QUASI-HEREDITARY ALGEBRAS

Piroska Lakatos

Institute of Mathematics and Informatics,
Lajos Kossuth University
P.O.Box 12, 4010 Debrecen, Hungary

Abstract. Inspired by the work of Mirollo and Vilonen [MV] describing the categories of perverse sheaves as module categories over certain finite dimensional algebras, Dlab and Ringel introduced [DR2] an explicit recursive construction of these algebras in terms of the algebras $A(\gamma)$. In particular, they characterized the quasi-hereditary algebras of Cline-Parshall-Scott [PS] and constructed them in this way. The present paper provides a characterization of lean algebras and some other special classes of algebras in terms of this recursive process.

1. Basics and Notation

The aim of this paper is to clarify the structure of particular types of quasi-hereditary algebras which appear in the applications to the theory of Lie algebras. Our method is based on the construction described in [DR2], taking into account some characteristic properties of the bimodules and maps which define the recursive process to build up quasi-hereditary algebras of certain type.

Let A be a basic finite dimensional K-algebra and $\mathbf{e} = \mathbf{e}_A = (e_1, e_2, e_3, \dots, e_n)$ a complete sequence of primitive orthogonal idempotents of the algebra A so that $\sum_{i=1}^{n} e_i = 1 : A_A = \overset{n}{\underset{i=1}{\oplus}} e_i A$. Write $\varepsilon_i = e_i + e_{i+1} + \ldots + e_n$ for $1 \le i \le n$ and $\varepsilon_{n+1} = 0$. Let us recall the definition of the right and left standard modules of A: $\Delta(i) = \Delta_A(i) = e_iA/e_iA\varepsilon_{i+1}A$ and $\Delta^o(i) = \Delta^o_A(i) = Ae_i/A\varepsilon_{i+1}Ae_i$,

Research partially supported by Hungarian NFSR grant No. TO16432.

respectively. A standard module is said to be Schurian if its endomorphism algebra is a division algebra. In general, write $d_i = \dim_K \operatorname{End} S(i)$. In what follows, $S(i)$ denote the simple right A-modules, $P(i) \simeq e_i A$ their projective covers and $V(i)$ the kernels of the canonical epimorphisms $P(i) \to \Delta(i)$ (see [D1] for the basic definitions and notation). Thus, for every $1 \le i \le n$ we have short exact sequences

$$0 \to V(i) \to P(i) \to \Delta(i) \to 0 \qquad \text{and} \qquad 0 \to U(i) \to \Delta(i) \to S(i) \to 0.$$

Of course, for the left modules there are similar canonical short exact sequences

$$0 \to V^o(i) \to P^o(i) \to \Delta^o(i) \to 0 \qquad \text{and} \qquad 0 \to U^o(i) \to \Delta^o(i) \to S^o(i) \to 0.$$

Given a (right) A-module X, define its *trace filtration* (with respect to $\mathbf{e}$) by

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \ldots \supseteq X^{(i)} \supseteq X^{(i+1)} \supseteq \ldots \supseteq X^{(n)} \supseteq X^{(n+1)} = 0,$$

where $X^{(i)}$ is the submodule of X generated by the homomorphic images of the module $\varepsilon_i A$ for $1 \le i \le n$. Considering the trace filtration of the algebra A

$$A = A\varepsilon_1 A \supseteq A\varepsilon_1 A \supseteq \ldots \supseteq A\varepsilon_i A \supseteq \ldots \supseteq A\varepsilon_n A \supseteq A\varepsilon_{n+1} A = 0,$$

we obtain its filtration by the idempotent ideals $A^{(i)} = A\varepsilon_i A$.

An algebra $(A, \mathbf{e})$ is said to be *quasi-hereditary* if, for all $1 \le i \le n$, the modules $\Delta(i)$ are Schurian and $(A\varepsilon_i A)\big/(A\varepsilon_{i+1} A) \simeq \oplus \Delta(i)$ (cf. the original definition of Cline-Parshall-Scott; see also [DR1]) .

Equivalently, the algebra $(A, \mathbf{e})$ is quasi-hereditary if

$$\dim_K A = \sum_{i=1}^{n} (1/d_i) \dim_K \Delta(i) \dim_K \Delta^o(i), \tag{1}$$

The equality (1) is equivalent to the fact that $\operatorname{End} \Delta(i) \simeq \operatorname{End} S(i)$ for all $1 \le i \le n$ and the regular representation A_A has a $\Delta-$filtration i.e there is a chain of submodules

$$A_A = X^{(0)} \supset \ldots \supset X^{(t)} \supset X^{(t+1)} \supset \ldots \supset X^{(l-1)} \supset X^{(l)} = 0$$

such that $X^{(t)}/X^{(t+1)} = \Delta(i_t)$ for all $0 \leq t \leq l-1$. Indeed, using an induction argument, this follows from the following statements $(A)-(C)$ (cf. [D2])

(A) For every (right) A-module X, $[X : S(i)] = (1/d_i)\dim_K Xe_i$; thus

$$[A_A : S(n)] = (1/d_n)\dim_K Ae_n = (1/d_n)\dim_K \Delta^o(n).$$

(B) Always, $\dim_K Ae_nA \leq (1/d_n)\dim_K \Delta(n)\dim_K \Delta^o(n)$.

(C) The equality $\dim_K Ae_nA = (1/d_n)\dim_K \Delta(n)\dim_K \Delta^o(n)$ holds if and only if $\operatorname{End}_A \Delta(n) = \operatorname{End}_A S(n)$ and $Ae_nA \simeq \underset{\text{finite}}{\oplus} \Delta(n)$.

2.The construction of $A(\gamma)$

Recall the (recursive) construction of quasi-hereditary algebra introduced in [DR2]. We shall modify it to describe a construction of particular classes of quasi-hereditary algebras.

Let D be a division K-algebra, C a basic K-algebra, ${}_DS_C$ and ${}_CT_D$ finite-dimensional bimodules with K acting centrally. Let $\gamma :\ {}_CT_D \otimes_D S_C \to {}_CC_C$ be a bimodule homomorphism whose image lies in $\operatorname{rad} C$. Let

$$B = D \ltimes ({}_DS_C \otimes_C T_D)$$

the "split" K-algebra with the coordinate-wise addition and multiplication given by

$$(d_1, s_1 \otimes t_1)(d_2, s_2 \otimes t_2) = \big(d_1d_2, d_1s_2 \otimes t_2 + s_1 \otimes t_1d_2 + s_1\gamma(t_1 \otimes s_2) \otimes t_2\big).$$

Clearly, B is a local K-algebra with $\operatorname{rad} B = S_C \otimes_C T$. It follows that S has the structure of a B-C-bimodule by $(d, s\otimes t)\cdot s' = ds' + s\gamma(t \otimes s')$ and T the structure of a C-B-bimodule by $t' \cdot (d, s \otimes t) = t'd + \gamma(t' \otimes s)t$. In [DR2], the 2×2 matrix $A = \begin{pmatrix} B & S \\ T & C \end{pmatrix}$ with multiplication given by

$$\begin{pmatrix} b & s \\ t & c \end{pmatrix}\begin{pmatrix} b' & s' \\ t' & c' \end{pmatrix} = \begin{pmatrix} bb' + (0, s\otimes t') & b\cdot s' + sc' \\ t\cdot b' + ct' & \gamma\big(t \otimes s'\big) + cc' \end{pmatrix}$$

is shown to be a $A(\gamma)$ ring, viz. the quotient of the tensor algebra over the $(C\times D)$-$(C\times D)$-bimodule $T \oplus S$ by the ideal

$$I(\gamma) = \langle t\otimes s - \gamma(t\otimes s) \mid t\in T, s\in S\rangle.$$

Note that $e_1 = \begin{pmatrix} (1,0) & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{e}_C = (e_2, e_3, \dots, e_n)$ is a complete sequence of primitive orthogonal idempotents of C. Dlab and Ringel have shown in [DR2] that $(A, \mathbf{e})$ is quasi-hereditary if and only if $(C, \mathbf{e}_C)$ is quasi-hereditary and S_C and ${}_CT$ have Δ_C-filtration and Δ^o_C-filtration, respectively; in fact, they have shown that all basic quasi-hereditary algebras over a perfect field K can be obtained by iterating this construction, starting with a division K-algebra C. Here, we are going to characterize lean algebras, as well as some special classes of quasi-hereditary algebras A in terms of properties of C, ${}_DS_C$, ${}_CT_D$ and the homomorphism γ.

Consider a quasi-hereditary algebra $(A, \mathbf{e})$ and the centralizer (quasi-hereditary) algebra (C, e_C), where $C = \varepsilon_2 A \varepsilon_2$, together with the C-modules $S_C = e_1 A \varepsilon_2$, and ${}_CT = \varepsilon_2 A e_1$. There is a close relationship between A and C given by the following pair of functors:

$$\Phi : \mathrm{mod}-A \rightarrow \mathrm{mod}-C \quad \text{and} \quad \Psi : \mathrm{mod}-C \rightarrow \mathrm{mod}-A$$

defined by $\Phi(X_A) = X\varepsilon_2$ and $\Psi(Y_C) = Y \underset{C}{\otimes} \varepsilon_2 A$. Recall that, for a Δ-filtered A-module X, the multiplication map

$$X\varepsilon_2 \underset{C}{\otimes} \varepsilon_2 A \longrightarrow X\varepsilon_2 A \ \text{ is } \quad \text{bijective} \ \ (\text{see[D1]}).$$

It follows that,for $i \geq 2$, $V_A(i)\varepsilon_2 = V_C(i)$ and $V_C(i)\varepsilon_2 \underset{C}{\otimes} \varepsilon_2 A = V_A(i)$, $P_A(i)\varepsilon_2 = P_C(i)$, $P_C(i) \underset{C}{\otimes} \varepsilon_2 A = P_A(i)$, $\Delta_A(i)\varepsilon_2 = \Delta_C(i)$, and $\Delta_A(i)$ is a quotient of $\Delta_C(i) \underset{C}{\otimes} \varepsilon_2 A$. In particular, $V(i)$ is a projective A-module if and only if $V(i)\varepsilon_2$ is a projective C-module. Furthermore, since $\varepsilon_2 A =_C (T \oplus C)_A$, $V_A(1) = S_C \otimes ({}_CT \oplus C)_A$ is a projective A-module if and only if S_C is a projective C-module.

Let us remark that all precedings statements apply also to the left C-modules $V^o_C(i)$, $P^o_C(i)$, $\Delta^o_C(i)$ and the left A-modules $V^o_A(i)$, $P^o_A(i)$, $\Delta^o_A(i)$.

3.Lean Algebras

An algebra algebra $(A, \mathbf{e})$ is said to be *lean* with respect to the order $\mathbf{e}$ (see [ADL] or [D1]) if

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_m(\operatorname{rad} A)e_j \text{ for all } 1 \leq i, j \leq n,\ m = \min\{i, j\}. \quad (2)$$

Equivalently, $(A, \mathbf{e})$ is lean if the standard modules are Schurian and, for all $1 \leq i \leq n$, both $V(i)$ and $V^o(i)$ are top submodules of $\operatorname{rad} P(i)$ and $\operatorname{rad} P^o(i)$ respectively (see [ADL]). Recall that a submodule X is a *top submodule* of Y if $\operatorname{rad} X = X \cap \operatorname{rad} Y$. Furthermore, top filtration of a module Z is a filtration whose members are top submodules of Z.

Proposition 1. *$(A, \mathbf{e})$ is lean if and only if $(C, \mathbf{e}_C)$ is a lean algebra and* $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$.

Proof. Let $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$ and $(C, \mathbf{e}_C)$ be lean. We are going to show (2). The equiality (2) is trivially true if $i = 1$ or $j = 1$; for, in this case $m = 1$ and $\varepsilon_1 = 1$. Thus, let both $i \geq 2, j \geq 2$. Then

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A) \sum_{t=1}^{n} e_t(\operatorname{rad} A)e_j =$$

$$= \sum_{t \geq 2} e_i(\operatorname{rad} A)e_t(\operatorname{rad} A)e_j + e_i(\operatorname{rad} A)e_1(\operatorname{rad} A)e_j.$$

The second summand $e_i(\operatorname{rad} A)e_1(\operatorname{rad} A)e_j = e_iAe_1Ae_j$ satisfies

$$e_iAe_1Ae_j = e_i(e_iAe_1)(e_1Ae_j)e_j \subseteq e_i(\operatorname{Im}\gamma)e_j \subseteq e_i(\operatorname{rad} C)^2 e_j.$$

Moreover, the first summand can be rewritten as

$$\sum_{t \geq 2}^{n} e_i(\operatorname{rad} A)e_t(\operatorname{rad} A)e_j =$$

$$= \sum_{t \geq 2} e_i(\varepsilon_2(\operatorname{rad} A)\varepsilon_2)e_t(\varepsilon_2(\operatorname{rad} A)\varepsilon_2)e_j + \sum_{t \geq 2} e_i(\operatorname{rad} C)e_t(\operatorname{rad} C)e_j = e_i(\operatorname{rad} C)^2 e_j.$$

Thus, since C is lean and $m = min\{i, j\} \geq 2$,

$$e_i(\operatorname{rad} C)^2 e_j = e_i(\operatorname{rad} C)\varepsilon_m(\operatorname{rad} C)e_j = e_i(\operatorname{rad} A)\varepsilon_m(\operatorname{rad} A)e_j,$$

as required.

Conversely, if A is lean, then C is obviously lean. Moreover

$$\operatorname{Im}\gamma = (\varepsilon_2 A e_1)(e_1 A \varepsilon_2) \subseteq \varepsilon_2(\operatorname{rad} A)e_1(\operatorname{rad} A)\varepsilon_2 \subseteq \varepsilon_2(\operatorname{rad} A)^2\varepsilon_2 = (\operatorname{rad} C)^2.$$

The prof is completed.

Recall that the quasi-hereditary algebra $(A, \mathbf{e})$ is said to be *replete* if all $V(i) = e_i A \varepsilon_{i+1} A$ are projective top submodules of $\operatorname{rad} P(i) = e_i(\operatorname{rad} A)$, and all $V^o(i) = A\varepsilon_{i+1} A e_i$ are projective top submodules of $\operatorname{rad} P^o(i) = (\operatorname{rad} A)e_i$ (see [ADL]). If $(A, \mathbf{e})$ is a replete quasi-hereditary algebra, then $(C, \mathbf{e}_C)$ is a replete quasi-hereditary algebra and both S_C and ${}_C T$ are projective C-modules. The following simple example shows that these conditions alone do not imply that $(A, \mathbf{e})$ is replete.

Example 1.

Let $(A, \mathbf{e})$ be the path algebra of the quiver $2 \longrightarrow 1 \longrightarrow 3$; then $(A, \mathbf{e})$ is quasi-hereditary (in fact, hereditary); the regular representations are as follows:

$$A_A = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \end{smallmatrix}, \quad {}_A A = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}.$$

Here, $(C, \mathbf{e}_C)$ is replete and both S_C and ${}_C T$ are (simple) projective C-modules, but $(A, \mathbf{e})$ is not replete. Notice that $e_2(\operatorname{rad} A)^2 e_3 \neq 0$ and $e_2(\operatorname{rad} A)e_3(\operatorname{rad} A)e_3 = 0$. Indeed, the missing property is leanness .

Proposition 2. *The algebra $(A, \mathbf{e})$ is a replete quasi-hereditary algebra if and only if $(C, \mathbf{e}_C)$ is a replete quasi-hereditary algebra, S_C and ${}_C T$ are projective C-modules and $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$.*

Proof. If $(A, \mathbf{e})$ is replete (and thus lean), one can see immediately that $(C, \mathbf{e}_C)$ is replete, S_C and ${}_C T$ projective and, in view of Proposition 1, $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$. Indeed, since $V_A(i)$ is a top submodule of $\operatorname{rad} P_A(i)$, for all $i \geq 2$, $V_C(i) = V_A(i)\varepsilon_2$ is a projective top submodule of $\operatorname{rad} P_C(i) = \operatorname{rad} P_A(i)\varepsilon_2$.

In order to prove that the conditions are sufficient, we need to show that $V_A(i)$ is a projective top submodule of $\operatorname{rad} P_A(i)$ for $1 \geq i \geq n$. First, consider

$i \geq 2$. Since $V_C(i) = e_i(\operatorname{rad} A)\varepsilon_{i+1}A\varepsilon_2$ is a projective top C-submodule of the C-module $\operatorname{rad} P_C(i) = e_i(\operatorname{rad} A)\varepsilon_2$, $\quad V(i) = V_C(i) \underset{C}{\otimes} \varepsilon_2 A$ is a projective A-module.

Moreover, since by Proposition 1; A is lean, we have the equality

$$e_i(\operatorname{rad} A)^2 e_{i+1} = e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)\varepsilon_{i+1},$$

and thus one can identify the top of $V_A(i)$ in the top of $\operatorname{rad} P_A(i)$ with the top of $V_C(i)$. This yields a top embedding of $V_A(i)$ in $\operatorname{rad} P_A(i)$.

For $i = 1$, $V_A(1) = \operatorname{rad} P_A(1) = S_C \underset{C}{\otimes} \varepsilon_2 A$ is a projective A-module since S_C is a projective C-module; futhermore, $V_A(1)$ is obviously embedded in $\operatorname{rad} P_A(1)$ as a top submodule. One can use similar arguments to deal with the left A-modules $V_A^o(i)$ which completes the proof.

Recall that the quasi-hereditary algebra $(A, \mathbf{e})$ is called *shallow* if all $\operatorname{rad} \Delta(i)$ and $\operatorname{rad} \Delta^o(i)$ are semi-simple, $1 \leq i \leq n$. This is equivalent to the fact (see [ADL]) that

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j \text{ for all } 1 \leq i, j \leq n,\ M = \max\{i, j\}. \quad (3)$$

As a consequence, both Δ_C-filtration of S_C and Δ_C^o-filtration of ${}_CT$ are in this case top filtrations (see [ADL]), and $(C, \mathbf{e})$ is shallow. The above Example 1 shows that these properties are not sufficient for $(A, \mathbf{e})$ to be shallow. In order to obtain a characterization of shallow algebras we need again to guarantee leaness of A.

PROPOSITION 3. *$(A, \mathbf{e})$ is a shallow quasi-hereditary algebra if and only if $(C, \mathbf{e}_C)$ is a shallow quasi-hereditary algebra, S_C has a top Δ_C-filtration, ${}_CT$ has a top Δ_C^o-filtration and* $\operatorname{Im} \gamma \subseteq (\operatorname{rad} C)^2$.

Proof. We need only to show that the conditions for C, S_C, ${}_CT$ and γ are sufficient to imply (3).

For $i = j = 1$, there is nothing to prove. If $i = 1, j \geq 2$, then the Δ-filtraton of $\operatorname{rad} P(1)$ induced by the top Δ_C-filtration of S_C (which exists by [DR2]), is a top filtration. Consequently,

$$e_1(\operatorname{rad} A)^2 e_j \subseteq e_1(\operatorname{rad} A)\varepsilon_j(\operatorname{rad} A)e_j.$$

A similar argument works for $i \geq 2, j = 1$.

Hence, let $i \geq 2, j \geq 2$. Then, in view of the fact that $(C, \mathbf{e}_C)$ is shallow,

$$e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j = e_i(\operatorname{rad} C)\varepsilon_M(\operatorname{rad} C)e_j = e_i(\operatorname{rad} C)^2 e_j,$$

which, in turn equals to $e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)e_j$. By Proposition 1, $(A, \mathbf{e})$ is lean, and thus, for $m = min\{i, j\}$,

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_m(\operatorname{rad} A)e_j \subseteq e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)e_j,$$

as required.

The following two classes of lean algebras introduced in [ADL] (see also [D1]) fall in between the shallow and replete algebras. A quasi-hereditary algebra $(A, \mathbf{e})$ is called *right medial* if for every $1 \leq i \leq n$, $V(i)$ is a top submodule of $\operatorname{rad} P(i)$ and both $\operatorname{rad} \Delta(i)$ and $V(i)$ have top Δ-filtrations. Equivalently, $(A, \mathbf{e})$ is right medial, if for every $1 \leq i \leq n$, $V(i)$ is a top submodule of $\operatorname{rad} P(i)$ which has a top Δ-filtration and $V^o(i)$ is a projectiv top submodule of $\operatorname{rad} P^o(i)$. The algebra $(A, \mathbf{e})$ is called left medial if its opposite $(A^{op}, \mathbf{e})$ is right medial. Thus $(A, \mathbf{e})$ is left medial if, for every $1 \leq i \leq n$, $\operatorname{rad} \Delta(i)$ is semi-simple and $V(i)$ is a projective top submodule of $P(i)$ (see [ADL]). As a result, a characterization of right and left medial algebras, can be obtained by combining the conditions of Proposition 2 and 3.

Proposition 4. *The algebra $(A, \mathbf{e})$ is a right medial quasi-hereditary algebra if and only if $(C, \mathbf{e}_C)$ is a right medial quasi-hereditary algebra, S_C has a top Δ_C-filtration, ${}_CT$ is a projective C-module and* $\operatorname{Im} \gamma \subseteq (\operatorname{rad} C)^2$.

Proof. If $(A, \mathbf{e})$ is right medial, then for $i \geq 2$, both $\operatorname{rad} \Delta_C(i) = [\operatorname{rad} \Delta_A(i)]\varepsilon_2$ and $V_C(i) = V_A(i)\varepsilon_2$ have top Δ_C-filtrations. Thus $(C, \mathbf{e}_C)$ is right medial.

Moreover, $S_C = V_A(1)\varepsilon_2$ has a top Δ_C-filtration and ${}_CT = \varepsilon_2 V^o_A(1)$ is projective. Finally, since $(A, \mathbf{e})$ is lean, $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$ by Proposition 1.

Conversely, if the conditions for C, S_C, ${}_CT$ and γ are satisfied, the algebra $(A, \mathbf{e})$ is, by Proposition 1, a lean quasi-hereditary algebra. Thus, we can conclude that $V(1) = S_C \underset{C}{\otimes} \varepsilon_2 A$ has a top Δ-filtration and, for every $i \geq 2$, $V_A(i) = V_C(i) \underset{C}{\otimes} \varepsilon_2 A$ is a top submodule of $\operatorname{rad} P_A(i)$ with a top Δ-filtration. Furthermore, $V^o_A(1) = A\varepsilon_2 \otimes_C T$ is a projective top submodule of $\operatorname{rad} P^o_A(1)$ and, for every $i \geq 2$, $V^o_A(i) = A\varepsilon_2 \otimes V^o_C(i)$ is a projective top submodule of $\operatorname{rad} P^o_A(i)$. Consequently, $(A, \mathbf{e})$ is right medial.

Using the definition of left medial algebras, we get immediately the following characterization.

Proposition 4^{op}. *The algebra* $(A, \mathbf{e})$ *is a left medial quasi-hereditary algebra if and only if* $(C, \mathbf{e}_C)$ *is a left medial quasi-hereditary algebra,* S_C *is a projective* C*-module,* ${}_CT$ *has a top* Δ_C*-filtration, and* $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$.

The following example illustrates the situation.

Example 2.

Let A be the path algebra whose regular representations are as follows:

$$A_A = \begin{smallmatrix} & 1 & & & \\ 2 & & & 3 & \\ & & 1 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ & 3 & \\ 1 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 1 & & 2 \end{smallmatrix}, \quad {}_AA = \begin{smallmatrix} & 1 & \\ & 3 & \\ 1 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & & & \\ 1 & & & 3 & \\ & & 1 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 1 & & 2 \end{smallmatrix}.$$

Clearly, A is a right medial algebra which is not left medial. Here, the centralizer algebra is both right and left medial (in fact, shallow and replete), S_C has a top Δ_C-filtration, while T_C is a projective C-module (with a top Δ-filtration).

References

[ADL1] Ágoston, I., Dlab, V., Lukács, E., Lean quasi-hereditary algebras, *Representations of Algebras. Sixth International Conference, Ottawa, 1992.* CMS Conference Proceedings **14**, 1–14.

[D1] Dlab, V., Quasi-hereditary algebras, *Appendix to Drozd, Yu.A., Kirichenko, V.V., Finite dimensional algebras.* Springer-Verlag (1993.)

[D2] Dlab, V., Quasi-hereditary algebras revisited, *An. St.Univ. Ovidius Constantza,* **4** (1996), 43-54.

[DR1] Dlab, V., Ringel, C.M., Quasi-hereditary algebras, *Illinois J. of Math.* **35** (1989), 280–291.

[DR2] Dlab, V., Ringel, C.M., A construction for quasi-hereditary algebras, *Compositio Math.* **70** (1989), 155–175.

[MV] Mirollo, R., Vilonen, K., Berstein-Gelfand-Gelfand reciprocity on perverse sheaves, *Ann.Scient.Ec.Norm.Sup.* **20** (1987), 311–324.

[PS] Parshall, B.J., Scott, L.L., Derived categories, quasi-hereditary algebras, and algebraic groups, *Proc. Ottawa–Moosonee Workshop*, Carleton–Ottawa Math. Lecture Note Series **3** (1988), 1–105.